Problem 1. A line of uniform charge density λ lies along the x-axis as shown.

a. Find the total charge of the distribution.

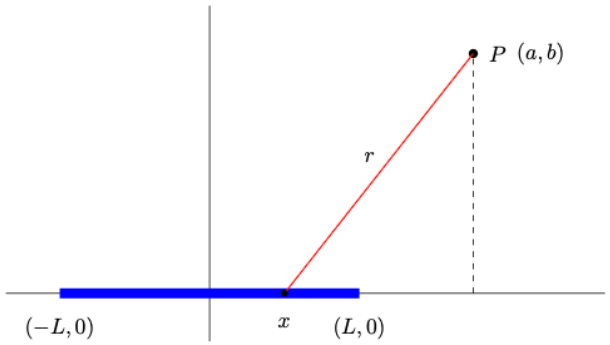
b. Find the electric field at point P by integrating the distribution over x. Show your work.

c. Now consider the special case where P lies directly above the centre of the rod, so that $a = 0$.

i. Simplify your answer from part (b) for this case. What happens to the x-component?

ii. Show that when $b \gg L$ your simplified result reduces to the field of a point charge, and state what that charge is.

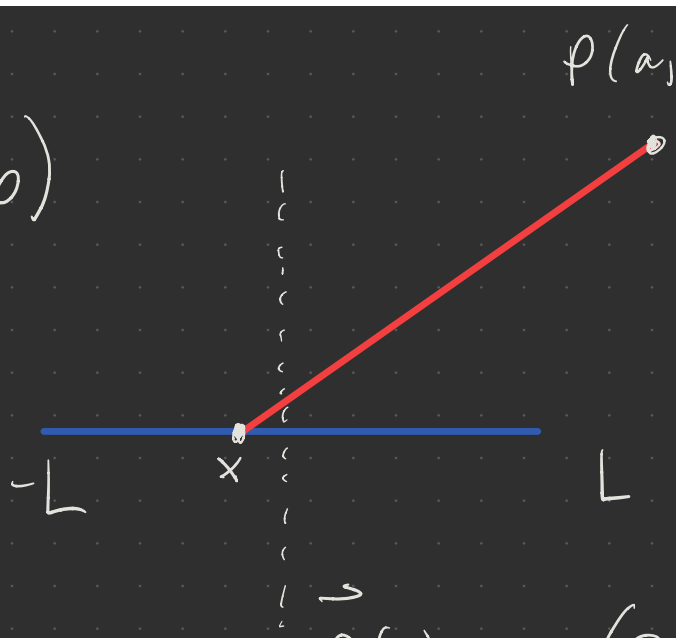
iii. Now take the *opposite* limit, $b \ll L$. What does your result become and how does the way the field falls off with b differ from part (ii)?



1a) $Q_{tot} = \lambda(2L)$

$= 2\lambda L$

b)



The plan will be to integrate x from −L to L. We need equation for r(x) in this range:

$\vec{r}(x) = (a-x)\hat{i} + b\hat{j}$ will work

$\vec{r} = (a-x)\hat{i} + b\hat{j}$ // Consider a small section of charge associated with the element dx

$dq = \lambda\, dx$

$\vec{r}/r^3 = \hat{r}/r^2$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{(a-x)\hat{i} + b\hat{j}}{((a-x)^2 + b^2)^{3/2}}\, \lambda\, dx$$

(dq)

Now evaluate the integral

there are two different terms and we must evaluate both!

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{(a-x)}{((a-x)^2+b^2)^{3/2}}\,dx$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(a-L)^2+b^2}} - \frac{1}{\sqrt{(a+L)^2+b^2}}\right)$$

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{b}{((a-x)^2+b^2)^{3/2}}\,dx$$ evaluating $\Rightarrow$

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\frac{1}{b}\left(\frac{L-a}{\sqrt{(a-L)^2+b^2}} + \frac{L+a}{\sqrt{(L+a)^2+b^2}}\right)$$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left[\left(\frac{1}{\sqrt{(a-L)^2+b^2}} - \frac{1}{\sqrt{(a+L)^2+b^2}}\right)\hat{i}\right.$$

$$\left. + \frac{1}{b}\left(\frac{L-a}{\sqrt{(a-L)^2+b^2}} + \frac{L+a}{\sqrt{(a+L)^2+b^2}}\right)\hat{j}\right]$$

c) i) Consider $a = 0$:

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left[\left(\frac{1}{\sqrt{L^2+b^2}} - \frac{1}{\sqrt{L^2+b^2}}\right)\hat{i} + \right.$$

$$\frac{1}{b}\left(\frac{L}{\sqrt{L^2+b^2}}+\frac{L}{\sqrt{L^2+b^2}}\right)\hat{j}\Bigg]$$

$$\therefore \quad \vec{E}=\frac{2L\lambda}{4\pi\epsilon_0 b}\,\frac{1}{\sqrt{L^2+b^2}}\,\hat{j}$$

$$=\frac{1}{4\pi\epsilon_0}\underbrace{(2L\lambda)}_{Q_{tot}}\frac{1}{b}\,\frac{1}{\sqrt{L^2+b^2}}\,\hat{j}$$

by symmetry the x-component of the $\vec{E}$-field cancels to zero.

c) ii) $$\vec{E} = \frac{1}{4\pi\epsilon_0}(2L\lambda)\frac{1}{b}\frac{1}{\sqrt{L^2+b^2}}\hat{j}$$

for $b \gg L$ term $$\frac{1}{\sqrt{L^2+b^2}} \approx \frac{1}{b}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}(2L\lambda)\frac{1}{b^2}\hat{j}$$

$1/r^2$ fall off of $\vec{E}$ field as expected.

c) iii) $\vec{E} = \frac{1}{4\pi\epsilon_0}(2L\lambda)\frac{1}{b}\frac{1}{\sqrt{L^2+b^2}}\hat{y}$

for $b \ll L$ $\frac{1}{\sqrt{L^2+b^2}} \simeq \frac{1}{L}$

$\vec{E} = \frac{1}{4\pi\epsilon_0}(2L\lambda)\frac{1}{b}\frac{1}{L}\hat{y}$

$= \frac{1}{4\pi\epsilon_0}\left(\frac{2\lambda}{b}\right)\hat{y}$

field looks like one from an infinite conductor, falls off $\propto 1/b$, not $1/b^2$

Problem 2. Point charges $Q_1 = +2.0\,\mu\text{C}$ and $Q_2 = -3.0\,\mu\text{C}$ are located at

$$\vec{r}_1 = (5.0\hat{\imath} + 1.0\hat{\jmath} + 3.0\hat{k})\,\text{m}$$

and

$$\vec{r}_2 = (3.0\hat{\imath} - 2.0\hat{\jmath} - 3.0\hat{k})\,\text{m}.$$

What is the force of Q_2 on Q_1? Give both the magnitude and write a unit vector (using $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$) to express the direction of the force.

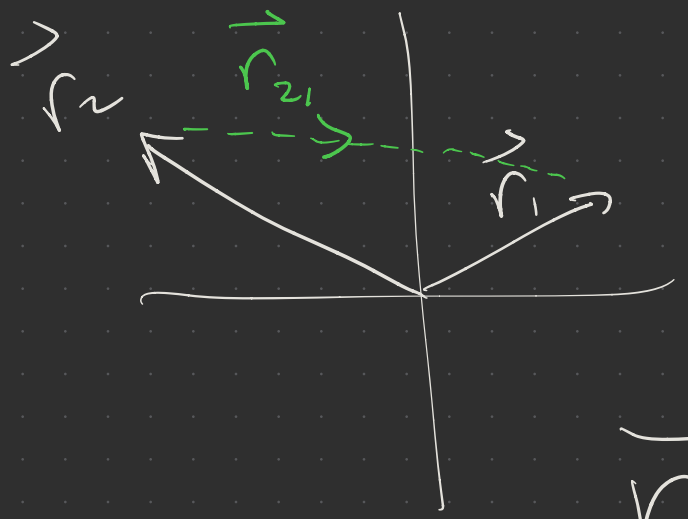
$\vec{r}_1$ and $\vec{r}_2$ are positions of charges wrt the origin.

We must find the force of Q_2 on Q_1: $\vec{F}_{21}$

We will need vector

$\vec{r}_{21}$

In general for $\vec{r}_1$ and $\vec{r}_2$ measured from the origin:



$$\vec{r}_2 + \vec{r}_{21} = \vec{r}_1$$

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$$

$$\vec{r}_{21} = (5\hat{\imath} + \hat{\jmath} + 3\hat{k}) - (3\hat{\imath} - 2\hat{\jmath} - 3\hat{k})$$

$$= (2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})\ m$$

$$|\vec{r}_{21}| = \sqrt{2^2 + 3^2 + 6^2} = (7)\ m$$

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{(+2\times10^{-6})(-3\times10^{-6})}{7^2} \frac{(2\hat{\imath}+3\hat{\jmath}+6\hat{k})}{7}$$

$$= 1.10\times10^{-3}\,N\left(-\frac{2}{7}\hat{\imath} - \frac{3}{7}\hat{\jmath} - \frac{6}{7}\hat{k}\right)$$

$$= 1.10\times10^{-3}\,N\left(-0.286\hat{\imath} - 0.429\hat{\jmath} - 0.857\hat{k}\right)$$

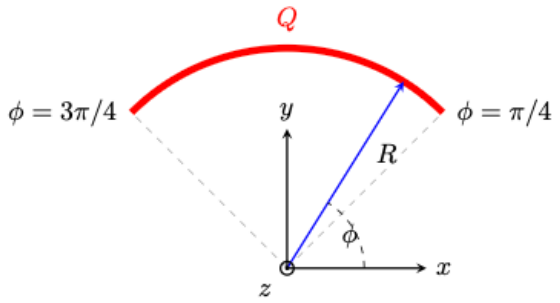
$$= \left(-0.314\hat{\imath} - 0.472\hat{\jmath} - 0.944\hat{k}\right)\times10^{-3}\,N$$

Problem 3. An arc of radius R has a total charge Q uniformly distributed along its length. The x-axis points to the right, the y-axis points to the top of the page, and the z-axis points toward you. The arc is centred on $(0,0,0)$ and extends from $\phi = \pi/4$ to $\phi = 3\pi/4$.

a. What is the linear charge density λ of the arc?

b. What is the electric field $\mathbf{E}(0,0,0)$ at the centre of the arc? Before integrating, state which component(s) of $\mathbf{E}$ must vanish and why.

c. What is the electric field $\mathbf{E}(0,0,z)$ along the axis of the arc, at a distance z above the page?



3a) Arc Length: $R\Delta\phi = R\left(\frac{3\pi}{4} - \pi/4\right)$

$\Delta l = R\,\pi/2$ $\quad \lambda = \frac{Q}{R\pi/2} = \frac{2Q}{\pi R}$

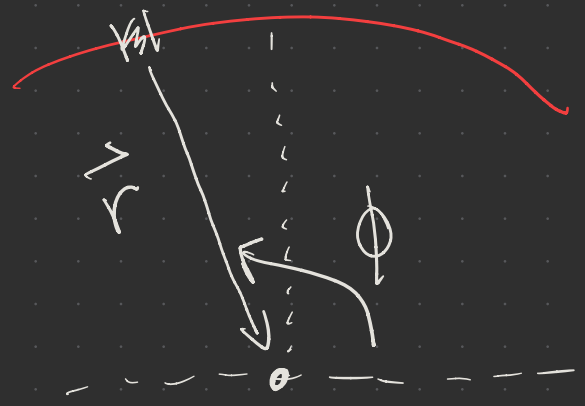
λ: Charge per unit length

b) x-component should vanish due to Symmetry

b) continued

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{r}}{r^3}\,dq$$

$dq = \lambda R\,d\phi$



$dq = \lambda R\,d\phi$

integrating ϕ from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$

$\vec{r} = -R\cos\phi\,\hat{i} - R\sin\phi\,\hat{j}$ (by inspection/check that this makes sense at diff values of ϕ)

Since Symmetry will make $E_x = 0$, we can just calculate

$$E_y = \frac{1}{4\pi\epsilon_0}\int_{\pi/4}^{3\pi/4} \frac{(-R\sin\phi)\,\lambda R\,d\phi}{\left((-R\cos\phi)^2 + (-R\sin\phi)^2\right)^{3/2}}$$

$$E_y = \frac{1}{4\pi\epsilon_0} \int_{\pi/4}^{3\pi/4} \frac{(-R\sin\phi)\,\lambda R\, d\phi}{\left((-R\cos\phi)^2 + (-R\sin\phi)^2\right)^{3/2}}$$

$$= \frac{+R^2\lambda}{4\pi\epsilon_0} \frac{\left(\cos\left(\frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{2}\right)\right)}{R^3}$$

$$= \frac{\left(-\sqrt{2}\right)\lambda}{4\pi\epsilon_0 R} = \frac{-\sqrt{2}\left(\frac{2Q}{\pi R}\right)}{4\pi\epsilon_0 R}$$

$$E_y = \frac{-\sqrt{2}\,Q}{2\pi^2\epsilon_0} \frac{1}{R^2}$$

3c) As we move along the z-axis our $\vec{r}$-vector goes from:

$$\vec{r} = -R\cos\phi\,\hat{\imath} - R\sin\phi\,\hat{\jmath}$$

to

$$\vec{r} = -R\cos\phi\,\hat{\imath} - R\sin\phi\,\hat{\jmath} + z\,\hat{k}$$

the axis of the arc is the z-axis

So we want $\vec{E}_z$

$$\vec{r} = -R\cos\phi\,\hat{i} - R\sin\phi\,\hat{j} + z\,\hat{k}$$

$$\vec{E}_z = \frac{1}{4\pi\epsilon_0}\int_{\pi/4}^{3\pi/4} \frac{z\,(\lambda R\,d\phi)}{\left((-R\cos\phi)^2 + (-R\sin\phi)^2 + z^2\right)^{3/2}}$$

$$= \frac{z\lambda R}{4\pi\epsilon_0}\int_{\pi/4}^{3\pi/4} \frac{d\phi}{\left(R^2 + z^2\right)^{3/2}}$$

$$= \frac{z\lambda R}{4\pi\epsilon_0} \int_{\pi/4}^{3\pi/4} \frac{d\phi}{(R^2+z^2)^{3/2}}$$

$$= \frac{z\lambda R}{4\pi\epsilon_0} \frac{1}{(R^2+z^2)^{3/2}} \left(\frac{\pi}{2}\right)$$

$$= \frac{z\left(\frac{2Q}{\pi R}\right) R \left(\pi/2\right)}{4\pi\epsilon_0 (R^2+z^2)^{3/2}}$$

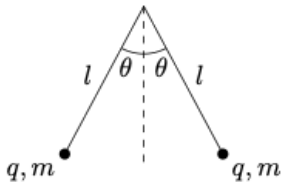
$$E_z = \frac{Q}{4\pi\epsilon_0} \frac{z}{(R^2+z^2)^{3/2}} \hat{k}$$

Problem 4. Each of the balls shown below carries a charge q and has a mass m. The length of each thread is l, and at equilibrium the balls are separated by an angle 2θ.

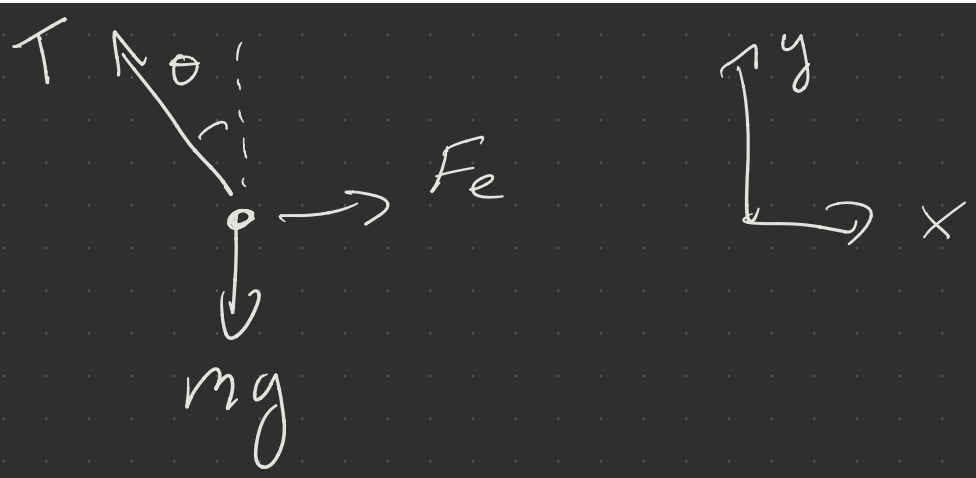
a. Draw a free-body diagram for one of the balls and show that θ satisfies

$$\sin^2\theta \tan\theta = \frac{q^2}{16\pi\epsilon_0 g l^2 m}.$$

b. Find q as a function of the measured angle θ (and m, l).



4a)



Along y: $T\cos\theta = mg$

Along x: $T\sin\theta = \dfrac{kq^2}{(2l\sin\theta)^2}$

$$T\cos\theta = mg$$

$$T\sin\theta = \frac{kq^2}{(2l\sin\theta)^2}$$

Eliminating T:

$$T = mg/\cos\theta$$

$$\left(\frac{mg}{\cos\theta}\right)\sin\theta = \frac{kq^2}{(2l\sin\theta)^2}$$

$$\sin^2\theta(\tan\theta) = \frac{kq^2}{4mgl^2} = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{4mgl^2}\right)$$

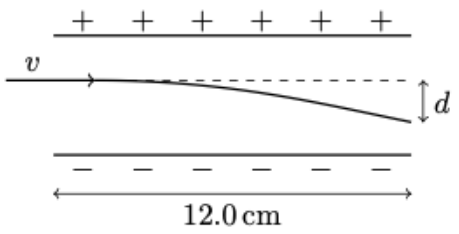
$$(\sin^2\theta)(\tan\theta) = \frac{q^2}{16\pi\epsilon_0 mgl^2}$$

Finally:

$$q = \sqrt{16\pi\epsilon_0 g l^2 m (\sin^2\theta \tan\theta)}$$

$$q = 4l\sin\theta\sqrt{\pi\epsilon_0 mg(\tan\theta)}$$

Problem 5. A proton enters the uniform electric field produced by the two charged plates shown below. The magnitude of the electric field is 4.0×10^5 N/C, and the speed of the proton when it enters is 1.5×10^7 m/s. What distance d has the proton been deflected downward when it leaves the plates?



$\vec{V_i} = 1.5 \times 10^7$ m/s



time to move across chamber:

$$t = \frac{d}{v} = \frac{0.12}{1.5 \times 10^7} = 8 \times 10^{-9} \text{ sec}$$

Force along $-y$ axis: $F = qE = (1.6 \times 10^{-19})(4 \times 10^5)$

$$= 6.4 \times 10^{-14} \text{ N}$$

acceleration: $\frac{F}{m} = a = \frac{6.4 \times 10^{-14} \text{ N}}{1.67 \times 10^{-27} \text{ kg}} = 3.83 \times 10^{13} \text{ m/s}^2$

Displacement in y-direction

$$\Delta y = -\frac{1}{2} a t^2$$

$$= -\frac{1}{2}\left(3.83 \times 10^{13}\ m/s^2\right)\left(8 \times 10^{-9}\ s\right)^2$$

$$= -1.23\ mm$$

$$= -1.23 \times 10^{-3}\ m$$

the proton is deflected by a distance of 1.23 mm downward