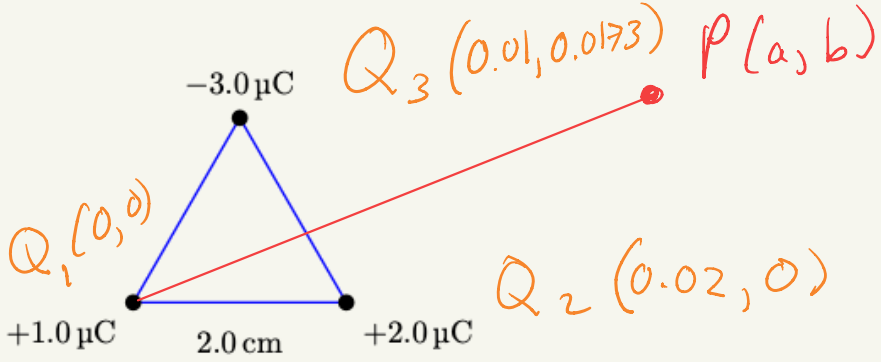
Problem 1. Three point charges, +1.0 µC, +2.0 µC, and −3.0 µC, sit at the corners of an equilateral triangle of side 2.0 cm, as shown below. Sketch the electric field of this charge distribution, showing enough lines to indicate the field's behaviour both close to each charge and far from the triangle. (*Hint:* What is the net charge of the distribution? How should that affect the field lines you draw at large distance?)



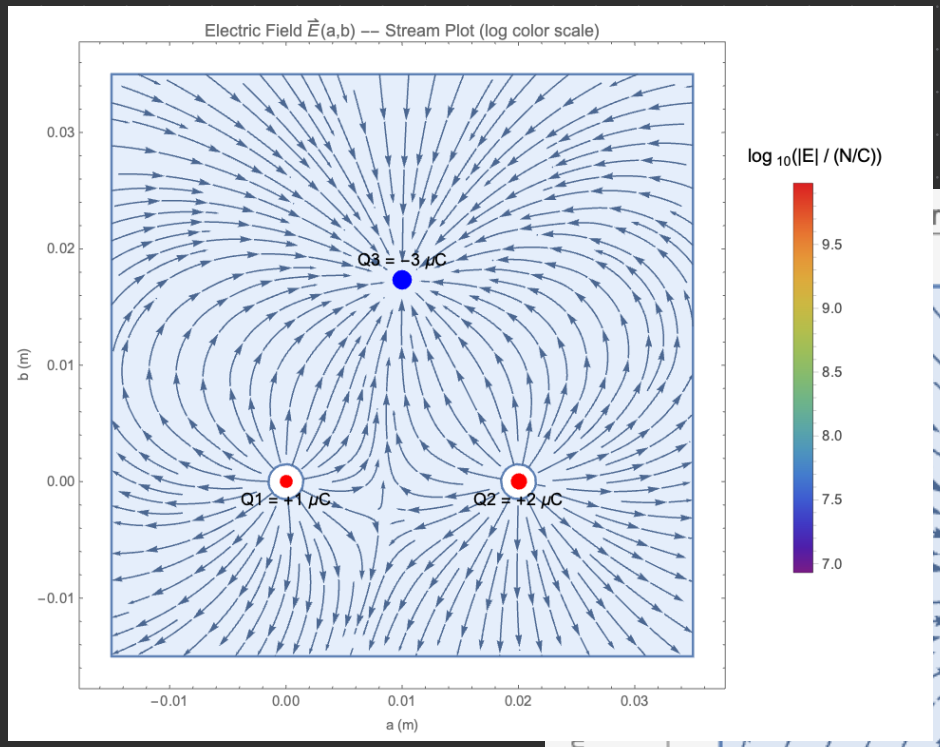


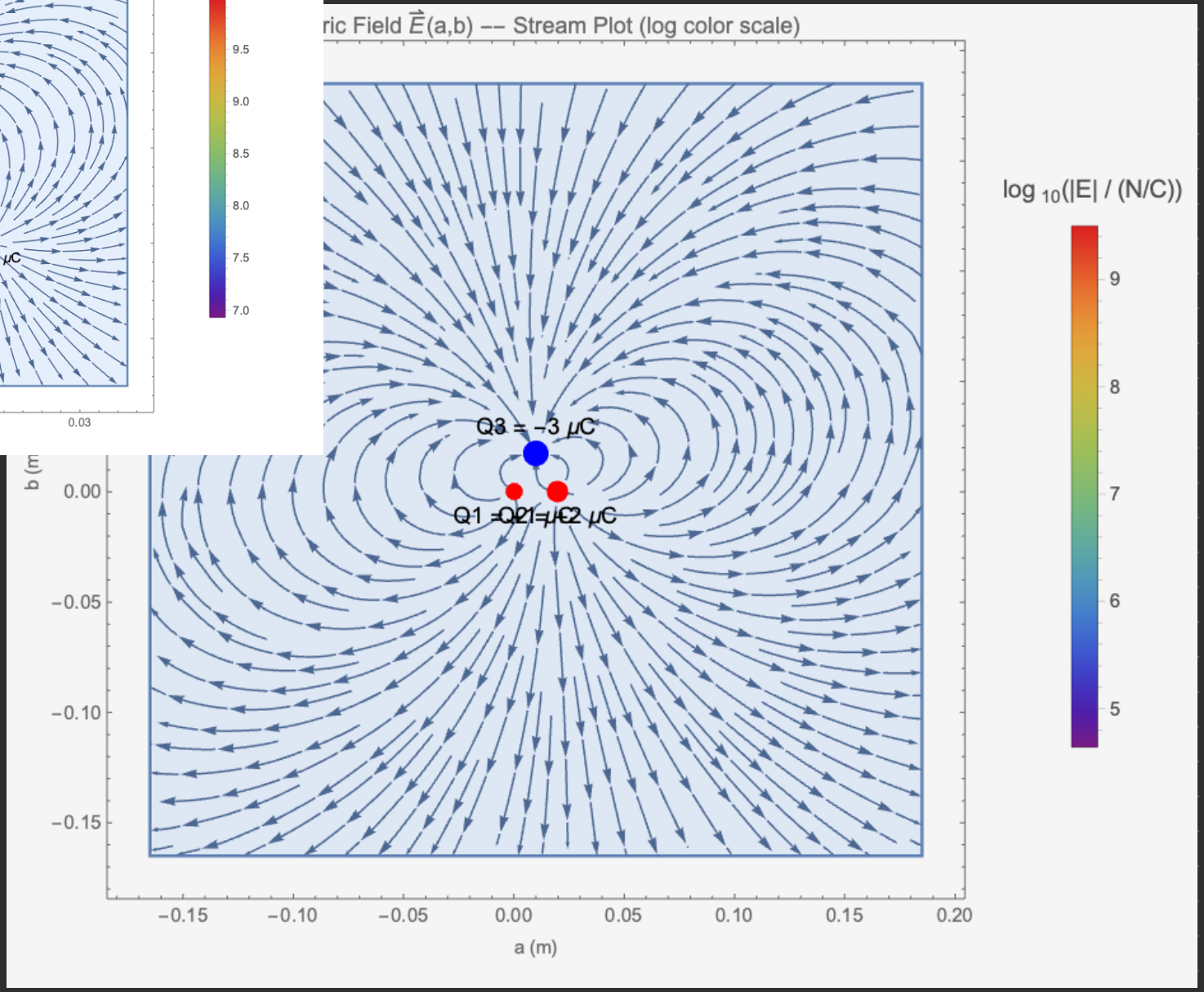
I'll choose origin at bottom left

$$\vec{r}_{1P} = a\hat{\imath} + b\hat{\jmath}, \quad \vec{r}_{2P} = (a - 0.02)\hat{\imath} + b\hat{\jmath}$$

$$\vec{r}_{3P} = \left((a - 0.01)\hat{\imath} + (b - 0.0173)\hat{\jmath}\right)$$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\left[Q_1 \frac{\vec{r}_{1P}}{|r_{1P}|^3} + Q_2 \frac{\vec{r}_{2P}}{|r_{2P}|^3} + Q_3 \frac{\vec{r}_{3P}}{|r_{3P}|^3}\right]$$





Since the net charge is zero (+1.0 - 3.0 + 2.0 = 0), the total flux through any Gaussian surface enclosing all three charges is zero, meaning the number of field lines leaving the surface must equal the number entering it.

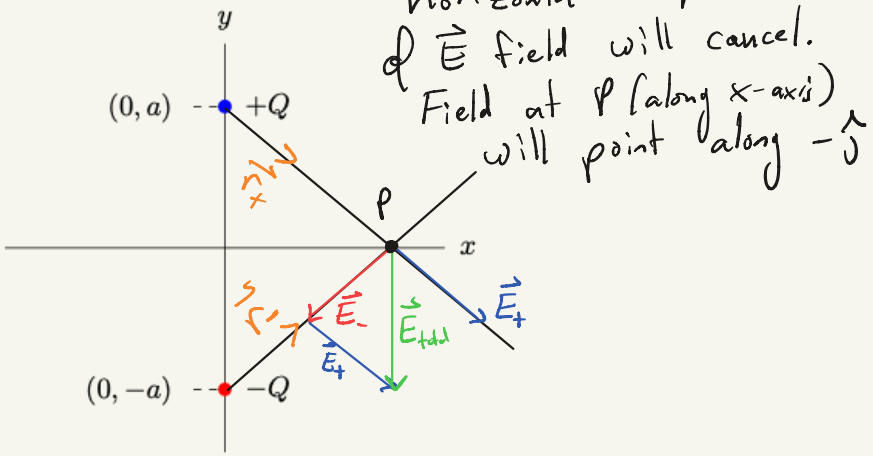
Problem 2. Two point charges $+Q$ and $-Q$ lie on the y-axis at $y = +a$ and $y = -a$, respectively, forming a dipole (see below). Let $Q = 4.0\,\mu\text{C}$ and $a = 1.0\,\text{mm}$.

a. Show that at all points on the x-axis for which $|x| \gg a$, $E \approx \dfrac{Qa}{2\pi\epsilon_0 x^3}$, and state the direction of $\vec{E}$ there.

b. Show that at all points on the y-axis for which $|y| \gg a$, $E \approx \dfrac{Qa}{\pi\epsilon_0 y^3}$.

c. What is the dipole moment $\vec{p}$ of this configuration? Give both magnitude and direction.

d. What is the torque $\vec{\tau}$ on this dipole in a uniform field $\vec{E} = 4.0 \times 10^5\,\text{N/C}\,\hat{\imath}$? Give both magnitude and direction.



horizontal components of $\vec{E}$ field will cancel. Field at P (along x-axis) will point along $-\hat{j}$

let P be at (x, y)

a) along x-axis $P(x, 0)$

$\vec{r}_+ = x\hat{\imath} - a\hat{\jmath}$

$\vec{r}_- = x\hat{\imath} + a\hat{\jmath}$

$\vec{E}_P = kQ\dfrac{\vec{r}_+}{|r_+|^3} - kQ\dfrac{\vec{r}_-}{|r_-|^3}$

$|\vec{r}_+| = |\vec{r}_-| = \sqrt{x^2 + a^2} = r$

$\vec{E}_P = \dfrac{kQ}{r^3}(x\hat{\imath} - a\hat{\jmath}) + \dfrac{kQ}{r^3}(-x\hat{\imath} - a\hat{\jmath})$

$= \dfrac{kQ}{r^3}(-2a\hat{\jmath}) = -\dfrac{2kQ}{r^3}a\hat{\jmath}$

$$\vec{E}_p = \frac{-2kQ}{r^3} a\,\hat{\jmath} \qquad k = \frac{1}{4\pi\epsilon_0} \Rightarrow \vec{E}_p = \frac{-Q}{2\pi\epsilon_0}\frac{a}{r^3}\hat{\jmath}$$

and for $|x| \gg a \qquad r = \sqrt{x^2 + a^2} \simeq x$

Along x-axis $\qquad |E| \simeq \frac{Qa}{2\pi\epsilon_0}\frac{1}{x^3}$

direction of $\vec{E}$ is downwards, $-\hat{\jmath}$ direction.

Along y-axis: $\quad P(0, y)$ $\qquad r_+ = (y-a)\hat{\jmath}$

$r_- = (y+a)\hat{\jmath}$

$$\vec{E}_p = kQ\frac{\vec{r}_+}{|r_+|^3} - kQ\frac{\vec{r}_-}{|r_-|^3}$$

$$\vec{E}_p = kQ\frac{\vec{r}_+}{|r_+|^3} - kQ\frac{\vec{r}_-}{|r_-|^3}$$

$$r_+ = (y-a)\hat{\jmath} \qquad |r_+| = |y-a|$$

$$r_- = (y+a)\hat{\jmath} \qquad |r_-| = |y+a|$$

for $y > a$

$$= \frac{kQ(y-a)\hat{\jmath}}{(y-a)^{3/2}} - \frac{kQ(y+a)\hat{\jmath}}{(y+a)^{3/2}}$$

$$= kQ\left(\frac{1}{(y-a)^2} - \frac{1}{(y+a)^2}\right)\hat{\jmath} = kQ\left(\frac{(y+a)^2 - (y-a)^2}{(y+a)^2(y-a)^2}\right)\hat{\jmath}$$

$$= kQ\left(\frac{(y+a)^2 - (y-a)^2}{((y+a)/(y-a))^2}\right)\hat{\jmath} = \frac{kQ\left((y+a)^2 - (y-a)^2\right)}{(y^2-a^2)^2}\hat{\jmath}$$

$$= \frac{kQ\left(y^2 + 2ay + a^2 - y^2 + 2ay - a^2\right)}{(y^2-a^2)^2}\hat{\jmath}$$

$$\vec{E}_p = \frac{kQ\,(4ay)\,\hat{j}}{(y^2 - a^2)^2}$$

Now for $|y| >> a$

$$\vec{E}_p \simeq \frac{kQ\,(4ay)}{y^4}\hat{j} = \frac{4kQa}{y^3}\hat{j}$$

$$= \frac{Qa}{\pi\epsilon_0}\frac{1}{y^3}\hat{j} \qquad |E_p| = \frac{Qa}{\pi\epsilon_0}\frac{1}{y^3}$$

direction is up

along $+\hat{j}$

c) Dipole moment: $\vec{p} = q\vec{d}$

$q = Q \qquad \vec{d} = 2a\hat{j}$

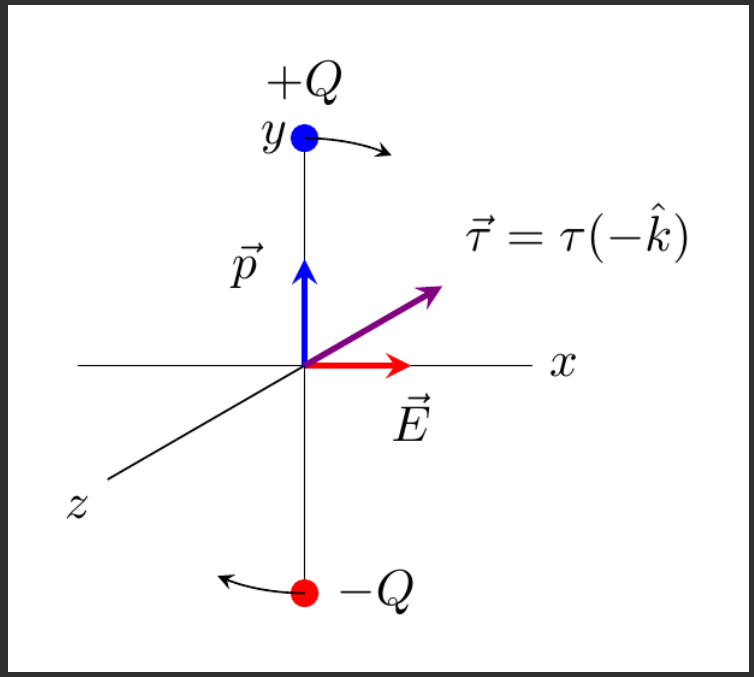
$\vec{p} = 2aQ\hat{j} = 2(1\times10^{-3}\,m)(4\times10^{-6}\,C)\hat{j}$

$\vec{p} = 8\times10^{-9}\,C\,m\,\hat{j}$

d) $\vec{\tau} = \vec{p}\times\vec{E} \qquad \vec{E} = (4\times10^{5}\,N/C)\hat{i}$

$\vec{p} = 8 \times 10^{-9}\ C\,m\ \hat{j}$

d) $\vec{\tau} = \vec{p} \times \vec{E}$ $\quad \vec{E} = (4 \times 10^{5}\ N/C)\,\hat{i}$




```python
import numpy as np

Q = 4.0e-6        # C
a = 1.0e-3        # m
E_mag = 4.0e5     # N/C

p = np.array([0, Q*(2*a), 0])   # dipole moment, points +y (from -Q to +Q)
E = np.array([E_mag, 0, 0])     # field, points +x
tau = np.cross(p, E)
print("p =", p, "C*m")
print("E =",E,"N/C")
print("tau =", tau, "N*m")
print("|tau| =", np.linalg.norm(tau), "N*m")
```

```
p = [0.e+00 8.e-09 0.e+00] C*m
E = [400000.      0.      0.] N/C
tau = [ 0.      0.     -0.0032] N*m
|tau| = 0.0032 N*m
```

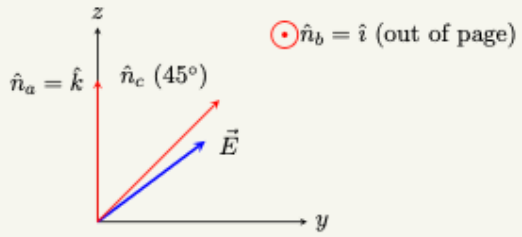
$\vec{\tau} = -3.2 \times 10^{-3}\ N\,m$

Problem 3. A uniform electric field $\vec{E} = (4.0\hat{\jmath} + 3.0\hat{k}) \times 10^3\,\mathrm{N/C}$ exists in a region of space. A flat circular area of radius 2.0 m is placed in this field. Find the electric flux through the area when it lies:

a. in the xy-plane (normal $\hat{n} = \hat{k}$);

b. in the yz-plane (normal $\hat{n} = \hat{\imath}$);

c. tilted 45° from the xy-plane by rotating about the x-axis, so that $\hat{n} = \sin(45°)\hat{\jmath} + \cos(45°)\hat{k}$ (see below).



$\vec{E} = (4\hat{\jmath} + 3\hat{k}) \times 10^3\ N/C$

a) $|\vec{A}| = \pi r^2 = 12.6\ m^2 \qquad \vec{A} = (12.6\hat{k})\ m^2$

$\Phi = \vec{E} \cdot \vec{A} = E_x A_x + E_y A_y + E_z A_z = 0 + (4\times10^3)(0) + (3\times10^3)(12.6)$

$= 3.77 \times 10^4\ N m^2/C$

b) $\Phi = \vec{E} \cdot (|A|\hat{\imath}) = 0$

$\Phi = (4\times10^3)\left(\frac{12.6}{\sqrt{2}}\right) + 3\times10^3\left(\frac{12.6}{\sqrt{2}}\right)$

c) $\vec{A} = (12.6\ m^2)\left(\frac{\hat{\jmath}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right)$

$= 6.24 \times 10^4\ N m^2/C$

Problem 4. Charge is distributed throughout a spherical shell of inner radius r_1 and outer radius r_2 with volume charge density

$$\rho(r) = \rho_0 \frac{r_1}{r}, \qquad r_1 \le r \le r_2,$$

and $\rho = 0$ elsewhere, where ρ_0 is a constant.

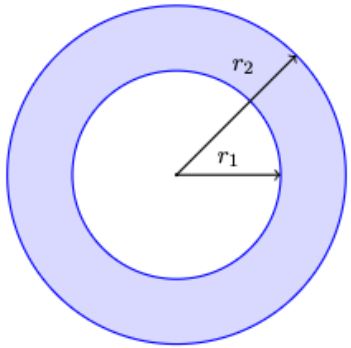
a. Find the total charge Q contained in the shell. (*Hint:* Because ρ varies with r, divide the shell into thin spherical layers of radius r and thickness dr, each with volume $dV = 4\pi r^2\, dr$, and integrate.)

Using Gauss's law, find $E(r)$ for

b. $r < r_1$;

c. $r_1 \le r \le r_2$;

d. $r > r_2$. Express your answer in terms of Q from part (a), and comment on the result.



a)

$$\rho(r) = \rho_0 \frac{r_1}{r}$$

$$dq = \rho\, dV$$

$$= \rho_0 \frac{r_1}{r}(4\pi r^2\, dr)$$

$$dq = 4\pi \rho_0 r_1\, r\, dr$$

$$Q = \int_{r_1}^{r_2} (4\pi \rho_0 r_1)\, r\, dr$$

$$= 4\pi \rho_0 r_1 \frac{r^2}{2}\Big|_{r_1}^{r_2}$$

$$= 4\pi \rho_0 r_1 \left(\frac{r_2^2 - r_1^2}{2}\right)$$

$$Q = 2\pi \rho_0 r_1 (r_2^2 - r_1^2)$$

b) $r < r_1$, $Q_{enc} = 0$

$$E(r) = 0$$

$$dq = 4\pi \rho_0 r_1 \, r \, dr$$

c) $$Q_{encl} = \int_{r_1}^{r} 4\pi \rho_0 r_1 \, r \, dr$$

Spherical Gaussian Surface

$$\Phi = \frac{Q_{encl}}{\epsilon_0} = EA = E(4\pi r^2)$$

$$E(r) = \left(\frac{1}{4\pi r^2}\right) \frac{1}{\epsilon_0} \int_{r_1}^{r} 4\pi \rho_0 r_1 \, r \, dr$$

↑ E as a function of r

NOT E times (r)

$$E(r) = \left(\frac{1}{4\pi r^2}\right) \frac{1}{\epsilon_0} \int_{r_1}^{r} 4\pi \rho_0 r_1 \, r \, dr$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \left(4\pi \rho_0 r_1 \int_{r_1}^{r} r \, dr \right)$$

$$= \frac{\rho_0 r_1}{\epsilon_0 r^2} \left. \frac{r^2}{2} \right|_{r_1}^{r}$$

$$= \frac{\rho_0 r_1}{\epsilon_0 r^2} \frac{r^2 - r_1^2}{2}$$

c) $$E(r) = \frac{\rho_0 r_1}{2\epsilon_0} \left(1 - \frac{r_1^2}{r^2}\right)$$

d) for $r > r_2$ all charge is enclosed

$$Q_{tot} = 2\pi \rho_0 r_1 (r_2^2 - r_1^2)$$

$$EA = \frac{Q_{tot}}{\epsilon_0} = E(4\pi r^2) = \frac{2\pi \rho_0 r_1 (r_2^2 - r_1^2)}{\epsilon_0}$$

$$E = \frac{1}{4\pi} \frac{1}{r^2} \frac{2\pi \rho_0 r_1}{\epsilon_0} (r_2^2 - r_1^2)$$

$$E(r) = \frac{\rho_0 r_1}{2\epsilon_0} (r_2^2 - r_1^2) \frac{1}{r^2}$$