P216 Assignment #6 — Solutions

Due Nov 4

Chapters 9–10: Current, Resistance & DC Circuits

Problem 1. Extension cord.

a. The cross-sectional area is $A = \pi(0.6455 \times 10^{-3}\,\mathrm{m})^2 = 1.31 \times 10^{-6}\,\mathrm{m}^2$, and the current travels $2 \times 30.0\,\mathrm{m} = 60.0\,\mathrm{m}$ of wire:

$$R = \rho\frac{L}{A} = (1.72 \times 10^{-8}\,\Omega\,\mathrm{m})\frac{60.0\,\mathrm{m}}{1.31 \times 10^{-6}\,\mathrm{m}^2} = \boxed{0.788\,\Omega}.$$

b.

$$\Delta V = IR = (10.0\,\mathrm{A})(0.788\,\Omega) = \boxed{7.88\,\mathrm{V}}, \qquad P = I^2R = (10.0\,\mathrm{A})^2(0.788\,\Omega) = \boxed{78.8\,\mathrm{W}}.$$

The heater receives $120 - 7.88 = \boxed{112\,\mathrm{V}}$. (This treats the current as fixed at 10.0 A. With the lower voltage, the heater's current and output also drop somewhat.)

c. For equal resistance at equal length, ρ/A must be the same, so $A \propto \rho$ and $d \propto \sqrt{\rho}$:

$$d_{\mathrm{Al}} = d_{\mathrm{Cu}}\sqrt{\frac{\rho_{\mathrm{Al}}}{\rho_{\mathrm{Cu}}}} = (1.291\,\mathrm{mm})\sqrt{\frac{2.65}{1.72}} = \boxed{1.60\,\mathrm{mm}}.$$

d. Nearly 80 W is dissipated along the cord. If the cord is long and thin, or coiled up so the heat cannot escape, its temperature rises. Copper's resistance increases with temperature, which adds to the heating, and the insulation can soften, melt, or start a fire. A heavier gauge (smaller R) or shorter cord reduces I^2R.

Problem 2. Incandescent bulb: hot and cold resistance.

a. At normal operation,

$$R_{\text{hot}} = \frac{V^2}{P} = \frac{(120\,\text{V})^2}{100\,\text{W}} = \boxed{144\,\Omega}, \qquad I = \frac{P}{V} = \boxed{0.833\,\text{A}}.$$

b. With $\Delta T = 2550\,^\circ\text{C} - 20\,^\circ\text{C} = 2530\,^\circ\text{C}$,

$$R_0 = \frac{R_{\text{hot}}}{1 + \alpha \Delta T} = \frac{144\,\Omega}{1 + (4.5 \times 10^{-3}\,^\circ\text{C}^{-1})(2530\,^\circ\text{C})} = \frac{144\,\Omega}{12.4} = \boxed{11.6\,\Omega}.$$

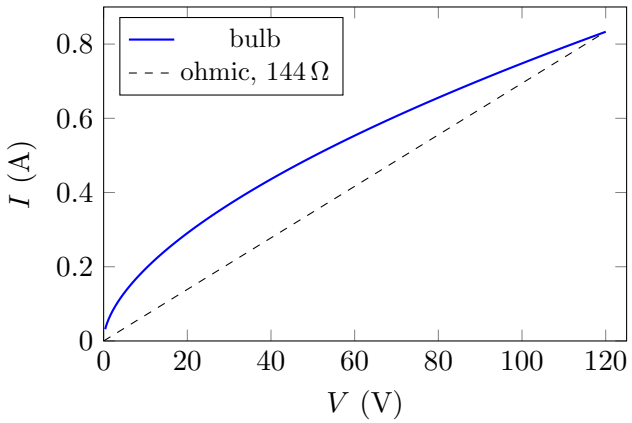
The hot filament has about 12 times the resistance of the cold one. (Measured ratios for real tungsten filaments are around 12–15; the linear model slightly underestimates the ratio because tungsten's resistivity rises faster than linearly at high temperature.)

c. At switch-on the full 120 V appears across the cold filament:

$$I_{\text{inrush}} = \frac{V}{R_0} = \frac{120\,\text{V}}{11.6\,\Omega} = \boxed{10.3\,\text{A}}, \qquad P_{\text{inrush}} = \frac{V^2}{R_0} = \boxed{1.24\,\text{kW}},$$

more than 12 times the rated power. The filament heats extremely rapidly (in tens of milliseconds) and the resistance rises until it settles at 144 Ω. Any thin spot in an aged filament (where tungsten has evaporated) has a higher local resistance per unit length, so it dissipates more power and heats faster than its surroundings during this surge; it overshoots and melts. That is why bulbs usually fail at the moment they are switched on, and why the thermal shock of each switch-on shortens filament life.

d. **No.** An ohmic device has $I \propto V$, i.e. R independent of V. Here R depends on temperature, which depends on the power and hence on V. At low voltage the filament is cool and R is small, so the I–V curve starts steeply; as V increases the filament heats, R grows, and the curve bends over (concave down). The ohmic resistor with $R = 144\,\Omega$ is a straight line through the origin that meets the bulb's curve only at 120 V; below 120 V the bulb always carries *more* current than that resistor. The curve below was generated with the linear $R(T)$ model and a filament cooled by radiation, $P \propto T^4$; any reasonable sketch with this shape gets full marks.



Problem 3. Jump-starting a car.

Throughout, $\varepsilon_1 = 12.6\,\text{V}$, $r_1 = 0.030\,\Omega$, $\varepsilon_2 = 11.0\,\text{V}$, $r_2 = 0.100\,\Omega$, $R_\text{s} = 0.060\,\Omega$. Take I_1 and I_2 as the currents *out of* the positive terminals of the two batteries, and I_s as the current down through the starter.

a. With S open there is a single loop through the two batteries. Their emfs oppose each other (positive to positive), so

$$I = \frac{\varepsilon_1 - \varepsilon_2}{r_1 + r_2} = \frac{1.6\,\text{V}}{0.130\,\Omega} = \boxed{12.3\,\text{A}},$$

flowing out of the good battery and *into* the positive terminal of the weak one, so the weak battery is being charged. The common terminal voltage is

$$V_t = \varepsilon_1 - Ir_1 = 12.6\,\text{V} - (12.3\,\text{A})(0.030\,\Omega) = \boxed{12.23\,\text{V}} \quad \big(\text{check: } \varepsilon_2 + Ir_2 = 12.23\,\text{V}\big).$$

$$P_\text{chem} = \varepsilon_2 I = (11.0\,\text{V})(12.3\,\text{A}) = \boxed{135\,\text{W}}, \qquad P_\text{heat} = I^2(r_1 + r_2) = (12.3\,\text{A})^2(0.130\,\Omega) = \boxed{19.7\,\text{W}}.$$

Energy check: the good battery supplies $\varepsilon_1 I = 155\,\text{W} = 135\,\text{W} + 19.7\,\text{W}$.

b. With S closed, the junction rule at the top node gives $I_\text{s} = I_1 + I_2$. Two loop equations (each battery around through the starter):

$$\varepsilon_1 - I_1 r_1 - I_\text{s} R_\text{s} = 0,$$
$$\varepsilon_2 - I_2 r_2 - I_\text{s} R_\text{s} = 0.$$

A tidy way to solve these: let $V = I_\text{s} R_\text{s}$ be the voltage across the starter. Then $I_1 = (\varepsilon_1 - V)/r_1$ and $I_2 = (\varepsilon_2 - V)/r_2$. Substituting into the junction rule $I_1 + I_2 = V/R_\text{s}$ gives

$$V = \frac{\varepsilon_1/r_1 + \varepsilon_2/r_2}{1/r_1 + 1/r_2 + 1/R_\text{s}} = \frac{420\,\text{A} + 110\,\text{A}}{(33.3 + 10.0 + 16.7)\,\Omega^{-1}} = \frac{530\,\text{A}}{60.0\,\Omega^{-1}} = \boxed{8.83\,\text{V}}.$$

$$I_1 = \frac{12.6\,\text{V} - 8.83\,\text{V}}{0.030\,\Omega} = \boxed{126\,\text{A}}, \qquad I_2 = \frac{11.0\,\text{V} - 8.83\,\text{V}}{0.100\,\Omega} = \boxed{21.7\,\text{A}}, \qquad I_\text{s} = \frac{8.83\,\text{V}}{0.060\,\Omega} = \boxed{147\,\text{A}}.$$

(Check: $126 + 21.7 = 147\,\text{A}$.) $I_2 > 0$, so the weak battery is now **discharging**: the starter pulls the terminal voltage down to 8.83 V, below ε_2, so the weak battery contributes current rather than absorbing it. (Direct matrix solution of the three equations gives the same result.)

c. Power to the starter:

$$P_\text{s} = I_\text{s}^2 R_\text{s} = \frac{V^2}{R_\text{s}} = \frac{(8.83\,\text{V})^2}{0.060\,\Omega} = \boxed{1.30\,\text{kW}}.$$

The weak battery alone would give

$$I = \frac{\varepsilon_2}{r_2 + R_\text{s}} = \frac{11.0\,\text{V}}{0.160\,\Omega} = 68.8\,\text{A}, \qquad P_\text{s} = I^2 R_\text{s} = \boxed{284\,\text{W}},$$

with only 4.1 V across the starter. The jump-start delivers about 4.6 times the power. The weak battery's real problem is its large internal resistance, not its slightly low emf: most of its power is wasted inside it. Energy check for (b): $\varepsilon_1 I_1 + \varepsilon_2 I_2 = 1582\,\text{W} + 238\,\text{W} = 1820\,\text{W}$, and $I_1^2 r_1 + I_2^2 r_2 + P_\text{s} = 473\,\text{W} + 47\,\text{W} + 1300\,\text{W} = 1820\,\text{W}$.

Problem 4. RC charging and discharging.

a.
$$\tau = RC = (47.0\times10^{3}\,\Omega)(220\times10^{-6}\,\mathrm{F}) = \boxed{10.3\,\mathrm{s}}, \qquad I_0 = \frac{\varepsilon}{R} = \frac{9.00\,\mathrm{V}}{47.0\,\mathrm{k\Omega}} = \boxed{191\,\mu\mathrm{A}}.$$
At $t = 0$ the uncharged capacitor has no voltage across it, so the full emf appears across R.

b. $V_C(t) = \varepsilon\big(1 - e^{-t/\tau}\big)$:

$$V_C(\tau) = (9.00\,\mathrm{V})(1 - e^{-1}) = \boxed{5.69\,\mathrm{V}}\ (63.2\%), \qquad V_C(2\tau) = (9.00\,\mathrm{V})(1 - e^{-2}) = \boxed{7.78\,\mathrm{V}}\ (86.5\%).$$

c. With $t/\tau = 10.0/10.34 = 0.967$ and $e^{-0.967} = 0.380$:

$$V_C = (9.00\,\mathrm{V})(1 - 0.380) = \boxed{5.58\,\mathrm{V}}, \qquad I = I_0 e^{-t/\tau} = (191\,\mu\mathrm{A})(0.380) = \boxed{72.8\,\mu\mathrm{A}}.$$
(Check: $V_R = IR = 3.42\,\mathrm{V}$, and $5.58 + 3.42 = 9.00\,\mathrm{V}$ ✓.)

d. Setting $1 - e^{-t/\tau} = \frac{1}{2}$ gives $e^{-t/\tau} = \frac{1}{2}$:
$$t_{1/2} = \tau \ln 2 = (10.34\,\mathrm{s})(0.693) = \boxed{7.17\,\mathrm{s}}.$$

e.
$$Q = C\varepsilon = (220\times10^{-6}\,\mathrm{F})(9.00\,\mathrm{V}) = \boxed{1.98\,\mathrm{mC}}, \qquad U = \tfrac{1}{2}C\varepsilon^2 = \boxed{8.91\,\mathrm{mJ}}.$$

f. The capacitor now discharges through the new resistor:
$$\tau' = (10.0\times10^{3}\,\Omega)(220\times10^{-6}\,\mathrm{F}) = \boxed{2.20\,\mathrm{s}}, \qquad I_0' = \frac{9.00\,\mathrm{V}}{10.0\,\mathrm{k\Omega}} = \boxed{0.900\,\mathrm{mA}},$$
$$V_C(5.00\,\mathrm{s}) = (9.00\,\mathrm{V})\,e^{-5.00/2.20} = (9.00\,\mathrm{V})(0.103) = \boxed{0.927\,\mathrm{V}}.$$
Discharge is faster than charge because the resistance is smaller.