PHYS 216 – Assignment #5 — Solutions – Due Oct 28

Chapters 8–9: Dielectrics, Current & Resistance

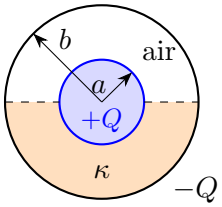
Problem 1. The spherical capacitor of Assignment 4, Problem 4 (inner sphere of radius a, thin outer shell of radius b, charges $\pm Q$) has capacitance $C_0 = 4\pi\varepsilon_0 ab/(b-a)$ when the space between the conductors is empty. Now the lower half of that space is filled with a liquid dielectric of dielectric constant κ, and the upper half is air (see below).

a. The field between the conductors is radial and has the same magnitude in both halves (you may assume this). Treat the arrangement as two hemispherical capacitors in parallel, one filled with dielectric and one with air, and find the capacitance C.

b. Find the potential difference between the conductors and the field $E(r)$ for $a < r < b$.

c. The free charge on the inner sphere is *not* uniformly distributed. Find the free surface charge density on the inner sphere in the air half and in the liquid half. What fraction of Q sits on the liquid half?

d. Find the bound surface charge density on the liquid where it touches the inner sphere. Show that the *total* (free plus bound) surface charge density is the same in both halves, and explain why this is consistent with the field being the same in both halves.



Solution 1

Notation: Q is free charge, Q_i (and σ_i) induced charge in the dielectric.

a. Each hemispherical half is half of the full spherical capacitor, so an empty half has $C_0/2$ and a filled half has $\kappa C_0/2$. They share the same ΔV (both connect the same two conductors), so they are in parallel:

$$C = \frac{C_0}{2} + \frac{\kappa C_0}{2} = \frac{1+\kappa}{2}\,C_0 = \frac{2\pi\varepsilon_0(1+\kappa)\,ab}{b-a}.$$

Check: $\kappa = 1$ recovers C_0.

b. $\Delta V = \dfrac{Q}{C} = \dfrac{Q(b-a)}{2\pi\varepsilon_0(1+\kappa)\,ab}$. The field is radial and the same in both halves, so $E = A/r^2$ (the field of a spherically symmetric total charge). Then $\Delta V = \int_a^b E\,dr = A\left(\frac{1}{a} - \frac{1}{b}\right) = A\,\frac{b-a}{ab}$, giving

$$E(r) = \frac{Q}{2\pi\varepsilon_0(1+\kappa)\,r^2}, \qquad a < r < b.$$

(For $\kappa = 1$ this is $Q/4\pi\varepsilon_0 r^2$, as it should be.)

c. At a conductor surface $D = \sigma_{\text{free}}$, i.e. $\sigma = \varepsilon_0 E$ against air and $\sigma = \kappa\varepsilon_0 E$ against the dielectric. At $r = a$:

$$\sigma_{\text{air}} = \varepsilon_0 E(a) = \frac{Q}{2\pi(1+\kappa)a^2}, \qquad \sigma_{\text{liq}} = \kappa\varepsilon_0 E(a) = \frac{\kappa Q}{2\pi(1+\kappa)a^2}.$$

Each half has area $2\pi a^2$, so the liquid half carries $Q_{\text{liq}} = \kappa Q/(1+\kappa)$ and the air half carries $Q/(1+\kappa)$; these add to Q. The fraction on the liquid half is $\kappa/(1+\kappa)$ (e.g. 80% for $\kappa = 4$). The free charge moves to the side where the dielectric partly cancels it.

d. The induced charge on the dielectric surface next to the inner sphere has the opposite sign, with $\sigma_i = -\left(1 - \frac{1}{\kappa}\right)\sigma_{\text{liq}}$:

$$\sigma_i = -\frac{(\kappa - 1)\, Q}{2\pi(1+\kappa)a^2}.$$

The total charge density in the liquid half is then

$$\sigma_{\text{liq}} + \sigma_i = \frac{\kappa Q - (\kappa - 1)Q}{2\pi(1+\kappa)a^2} = \frac{Q}{2\pi(1+\kappa)a^2} = \sigma_{\text{air}}.$$

The field is produced by *all* charge, free and induced. A Gaussian pillbox at the surface gives $E = \sigma_{\text{total}}/\varepsilon_0$, so equal total densities mean equal fields just outside the sphere. Equivalently, the total charge on the inner surface is uniform, with $Q_{\text{enc}} = 4\pi a^2\, \sigma_{\text{air}} = 2Q/(1+\kappa)$. Gauss's law with a sphere of radius r then gives $E = Q_{\text{enc}}/(4\pi\varepsilon_0 r^2) = Q/[2\pi\varepsilon_0(1+\kappa)r^2]$, which agrees with (b).

Problem 2. Two vertical parallel plates of width w (into the page) and separation d hang with their lower edges just touching the surface of a container of silicone oil (dielectric constant κ, density ρ). The plates are connected to a battery of potential difference V. The oil between the plates rises a height h above the level outside (measure h relative to the level with the battery disconnected, so surface tension can be ignored).

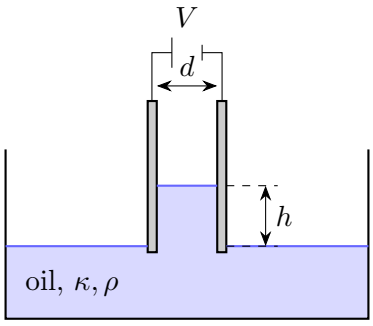
a. For plates of height H (from their lower edges, at the outside oil level), write the capacitance $C(h)$ as the sum of an oil-filled part and an air-filled part in parallel.

b. At constant V the upward force on the oil is $F = \frac{1}{2}V^2\, dC/dh$. Find F, and by balancing it against the weight of the raised oil show that

$$h = \frac{\varepsilon_0(\kappa - 1)V^2}{2\rho g d^2}.$$

c. Evaluate h for $\kappa = 2.7$, $\rho = 960\,\mathrm{kg/m^3}$, $d = 1.0\,\mathrm{mm}$ and $V = 2.5\,\mathrm{kV}$. The dielectric strengths of the oil and of air are $15\,\mathrm{MV/m}$ and $3.0\,\mathrm{MV/m}$. Is the capacitor safe from breakdown?

d. As the oil rises from 0 to h, the battery moves extra charge ΔQ onto the plates. Show that the work done by the battery, $V\Delta Q$, is twice the increase in stored energy. Where does the rest of the battery's energy go?



Solution 2

a. The oil-filled strip has height h and the air-filled strip has height $H - h$. Both have area (height)$\times w$ and gap d, and they are in parallel:

$$C(h) = \frac{\kappa\varepsilon_0 w h}{d} + \frac{\varepsilon_0 w(H-h)}{d} = \frac{\varepsilon_0 w}{d}\big[H + (\kappa - 1)h\big].$$

b. $dC/dh = \varepsilon_0 w(\kappa - 1)/d$, so

$$F = \tfrac{1}{2}V^2\frac{dC}{dh} = \frac{\varepsilon_0 w(\kappa-1)V^2}{2d},$$

which is independent of h. The raised oil is a slab of volume $w dh$, with weight $\rho g\, w dh$. Setting $F = \rho g\, w dh$:

$$h = \frac{\varepsilon_0(\kappa - 1)V^2}{2\rho g d^2}.$$

Note that w cancels.

c. Substituting the values,

$$h = \frac{(8.854 \times 10^{-12})(1.7)(2.5 \times 10^{3})^{2}}{2(960)(9.81)(1.0 \times 10^{-3})^{2}} = 5.0 \times 10^{-3}\,\mathrm{m} = 5.0\,\mathrm{mm}.$$

Between parallel plates $E = V/d$ in both the oil and the air, because the regions are in parallel and share V and d. So $E = (2.5\,\mathrm{kV})/(1.0\,\mathrm{mm}) = 2.5\,\mathrm{MV/m}$ everywhere. This is far below the oil's $15\,\mathrm{MV/m}$ and below the air's $3.0\,\mathrm{MV/m}$, so the capacitor is safe. The air region is the weak link: the margin there is only about 20%.

d. $\Delta Q = V\Delta C$ with $\Delta C = \varepsilon_0 w(\kappa - 1)h/d$. The battery does work $W_{\mathrm{bat}} = V\Delta Q = V^2\Delta C$. The stored energy at constant V rises by $\Delta U = \frac{1}{2}V^2\Delta C$, so $W_{\mathrm{bat}} = 2\Delta U$.

The other half, $\frac{1}{2}V^2\Delta C$, is the work done on the oil by the (constant) electric force: $Fh = \frac{1}{2}V^2(dC/dh)h$. Part goes into gravitational potential energy. The slab's centre of mass rises by $h/2$, so

$$\Delta U_g = \rho g\, wdh \cdot \frac{h}{2} = \frac{h}{2}\,F = \tfrac{1}{4}V^2\Delta C.$$

The remaining $\frac{1}{4}V^2\Delta C$ first becomes kinetic energy of the rising oil. Because F is constant while the weight grows linearly with height, the oil overshoots and oscillates about h. Viscosity (and some resistive loss in the leads) then dissipates it as heat. In summary: $\frac{1}{2}$ of the battery's work goes to field energy, $\frac{1}{4}$ to gravity, and $\frac{1}{4}$ to heat.

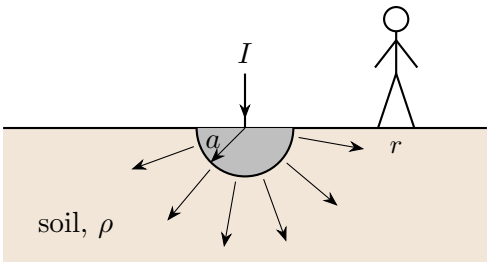
Problem 3. A manufacturing process produces hollow gold spheres with inner radius $a = 1.50\,\text{mm}$ and outer radius $b = 2.00\,\text{mm}$. The resistivity of gold is $\rho_0 = 2.44 \times 10^{-8}\,\Omega\,\text{m}$ at $T_0 = 20\,^\circ\text{C}$, with temperature coefficient $\alpha = 3.4 \times 10^{-3}\,^\circ\text{C}$.

a. Find a formula for the resistance between the inner and outer surfaces of a sphere at temperature T_0, and evaluate it. (Hint: add thin spherical shells of thickness dr in series.)

b. If the sphere is in an oven at $350\,^\circ\text{C}$, what is its resistance?

c. The same geometry appears in a grounding electrode at an electrical substation: a metal hemisphere of radius $a = 0.50\,\text{m}$ buried flat side up, flush with the surface, in soil of resistivity $\rho = 100\,\Omega\,\text{m}$. The soil now plays the role of the resistive material. The electrode is an ideal conductor ($0 \le r \le a$), so it holds the inner surface of the resistive "shell" at $r = a$ at a single potential, just as the inner surface did in (a). The distant earth plays the role of the outer surface, with $b \to \infty$. Adapt your formula from (a), accounting for the current spreading into only a half-space, to find the electrode's resistance to the distant earth.

d. During a fault, $I = 1.0\,\text{kA}$ flows into the ground through the electrode. The ground potential at distance r is $V(r) = I\rho/(2\pi r)$. A person stands with feet 1.0 m apart along a radial line, at $r = 5.0\,\text{m}$ and 6.0 m. Find the "step voltage" between their feet, and explain why a cow (front and hind legs about 1.5 m apart) is at greater risk.



Solution 3

a. A shell of radius r and thickness dr has length dr along the current and area $4\pi r^2$, so $dR = \rho_0\,dr/(4\pi r^2)$. The shells are in series:

$$R_0 = \frac{\rho_0}{4\pi}\int_a^b \frac{dr}{r^2} = \frac{\rho_0}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\rho_0(b-a)}{4\pi ab} = \frac{(2.44\times10^{-8})(0.50\times10^{-3})}{4\pi(1.50\times10^{-3})(2.00\times10^{-3})} = 3.24{\times}10^{-7}\,\Omega.$$

b. $R = R_0[1 + \alpha(T - T_0)] = (3.24\times10^{-7}\,\Omega)[1 + (3.4\times10^{-3})(330)] = (3.24\times10^{-7}\,\Omega)(2.12) = 6.87\times10^{-7}\,\Omega$. (Thermal expansion of the sphere is negligible by comparison.)

c. The soil is the resistor; the electrode is an equipotential at $r = a$ and the distant earth is the outer contact. The current now spreads through hemispherical shells of area $2\pi r^2$, not $4\pi r^2$, which doubles each dR. With $b \to \infty$:

$$R = \frac{\rho}{2\pi}\int_a^\infty \frac{dr}{r^2} = \frac{\rho}{2\pi a} = \frac{100\,\Omega\,\text{m}}{2\pi(0.50\,\text{m})} = 32\,\Omega.$$

(Consistent with the given $V(r)$: $V(a) = I\rho/2\pi a = IR$. At $I = 1.0\,\text{kA}$ the electrode rises about 32 kV above remote earth.)

d.

$$V_{\text{step}} = \frac{I\rho}{2\pi}\left(\frac{1}{5.0} - \frac{1}{6.0}\right) = \frac{(1.0\times 10^3)(100)}{2\pi}\left(\frac{1}{30}\right)\text{V} = 530\,\text{V}.$$

A cow at the same place, with front feet at 5.0 m and hind feet at 6.5 m, spans 1.5 m of the steep potential gradient: $V = \frac{I\rho}{2\pi}\left(\frac{1}{5.0} - \frac{1}{6.5}\right) \approx 730\,\text{V}$. It is at greater risk for two reasons. Its feet span more of the gradient ($dV/dr = I\rho/2\pi r^2$), and the current path from front to hind legs runs through the chest and heart. A person's foot-to-foot path passes mainly through the legs and lower abdomen.