# PHYS 216 – Assignment #4 – Solutions – Due Oct 21

Chapters 7–8: Potential, Conductors & Capacitance

Throughout, $\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C^2/N\,m^2}$.

**Problem 1.** **(a)** Take a coaxial Gaussian cylinder of radius $s$ and length $L$ ($a < s < R$). By symmetry $\vec{E}$ is radial and constant over the curved surface, and there is no flux through the ends. Only the wire's charge is enclosed:

$$E\,(2\pi s L) = \frac{\lambda L}{\varepsilon_0} \quad \Rightarrow \quad \boxed{E(s) = \frac{\lambda}{2\pi\varepsilon_0 s}} \text{ (radially outward).}$$

**(b)**

$$V_0 = V(a) - V(R) = \int_a^R E\,ds = \frac{\lambda}{2\pi\varepsilon_0}\int_a^R \frac{ds}{s} = \boxed{\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{R}{a}}.$$

**(c)** From (b), $\lambda/2\pi\varepsilon_0 = V_0/\ln(R/a)$. Substituting into (a):

$$\boxed{E(s) = \frac{V_0}{s\,\ln(R/a)}}.$$

**(d)** Here $\ln(R/a) = \ln 200 = 5.30$, so

$$E(a) = \frac{900\,\mathrm{V}}{(1.0\times10^{-4}\,\mathrm{m})(5.30)} \approx 1.7\times10^{6}\,\mathrm{V/m},$$

$$E(1.00\,\mathrm{cm}) \approx 1.7\times10^{4}\,\mathrm{V/m}, \qquad E(R) \approx 8.5\times10^{3}\,\mathrm{V/m}.$$

Because $E \propto 1/s$, the field at the wire is $R/a = 200$ times larger than at the wall. Only a thin sheath around the wire has a field approaching the breakdown scale ($E(a)$ is more than half of the $3\times10^{6}\,\mathrm{V/m}$ needed for dry air; real Geiger tubes use a low-pressure fill gas that breaks down at a lower field). Throughout the rest of the tube the field is two orders of magnitude below breakdown, so ionising avalanches start only next to the wire.

For a given $V_0$, $E(a) = V_0/[a\ln(R/a)]$ grows rapidly as $a$ shrinks (the ln varies only slowly). A rod of radius 1 cm would give only $E(a) = 900/(0.01\ln 2) \approx 1.3\times10^{5}\,\mathrm{V/m}$, far below breakdown. A thin wire produces a very large field with a modest, safe voltage. (This is the same physics as charge crowding at sharp points: $\sigma = \varepsilon_0 E$ is large on the thin wire.)

**(e)**

$$\frac{C}{L} = \frac{\lambda}{V_0} = \frac{2\pi\varepsilon_0}{\ln(R/a)} = \frac{2\pi(8.854\times10^{-12})}{5.30} = \boxed{10.5\,\mathrm{pF/m}}.$$

**Problem 2.** **(a)**

$$\boxed{C_0 = \frac{\varepsilon_0 A}{d}}, \qquad \boxed{C = \frac{\varepsilon_0 A}{d-t}}.$$

With the slab in place the field is zero inside the conductor, leaving two gaps $x_1 + x_2 = d - t$ in series: $1/C = x_1/\varepsilon_0 A + x_2/\varepsilon_0 A = (d-t)/\varepsilon_0 A$. Only the sum of the gaps enters, so the vertical position does *not* matter.

**(b)** The capacitor is isolated, so $Q$ is fixed:

$$\boxed{U_0 = \frac{Q^2 d}{2\varepsilon_0 A}}, \qquad \boxed{U = \frac{Q^2(d-t)}{2\varepsilon_0 A}}, \qquad \Delta U = -\frac{Q^2 t}{2\varepsilon_0 A} < 0.$$

The energy *decreases*: the field $Q/\varepsilon_0 A$ is unchanged in the gaps but has been removed from the volume $At$ occupied by the slab.

**(c)** The field points up, from the $+Q$ bottom plate to the $-Q$ top plate. Field lines leaving the bottom plate end on the slab's bottom face, and the lines ending on the top plate start on the slab's top face:

$$\boxed{\sigma_{\text{top face}} = +Q/A}, \qquad \boxed{\sigma_{\text{bottom face}} = -Q/A}.$$

The slab's total charge remains zero.

**(d)** *Sketch:* uniform vertical lines between the plates and the slab where the slab is in; uniform lines across the full gap where it is not; near the slab's leading edge the fringing lines bend to end on the slab's edge and corners. These fringing lines have a horizontal component at the slab's induced charges, pulling them (and so the slab) toward the region between the plates: the slab is *pulled in*. Equivalently, let $\ell$ be the distance the slab has been pushed in (not to be confused with the gaps $x_1$, $x_2$). At fixed $Q$ the energy decreases as $\ell$ increases, so $F = -dU/d\ell$ points inward. To slide it in slowly you must hold it back, so *you do negative work*:

$$W_{\text{you}} = \Delta U = -\frac{Q^2 t}{2\varepsilon_0 A}.$$

**(e)** Now the battery holds the potential difference fixed at its emf $V$, and $U = \frac{1}{2}CV^2$ *increases* because $C$ increases:

$$\Delta U = \tfrac{1}{2}(C - C_0)V^2 = \frac{\varepsilon_0 A V^2}{2}\,\frac{t}{d(d-t)} > 0.$$

The battery moves extra charge $\Delta Q = (C - C_0)V$ through potential difference $V$, doing work

$$W_{\text{batt}} = \Delta Q\, V = (C - C_0)V^2 = 2\,\Delta U.$$

Energy conservation, $W_{\text{batt}} + W_{\text{you}} = \Delta U$, gives $W_{\text{you}} = -\Delta U$: half the battery's work goes into stored energy, and the other half is work done by the field on the slab (taken up by your hand). The slab is still *pulled in* (at fixed $V$, $F = +dU/d\ell = \frac{1}{2}V^2\, dC/d\ell$, inward), and you still do negative work. The force depends on the local fields near the slab's edge, not on whether a battery is attached; the sign of $\Delta U$ differs only because the battery supplies energy.

**Problem 3.** **(a)** $C_{\text{eq}} = C_1 C_2/(C_1 + C_2) = 1.33\,\mu\text{F}$. In series both carry the same charge:

$$Q = C_{\text{eq}} V = \boxed{1.33\,\text{mC}} \text{ each}, \qquad V_1 = Q/C_1 = \boxed{667\,\text{V}}, \quad V_2 = Q/C_2 = \boxed{333\,\text{V}},$$

$$U_0 = \tfrac{1}{2}C_{\text{eq}}V^2 = \boxed{0.667\,\text{J}}.$$

**(b)** Joining like plates puts the two positive plates together, so the total charge on that side is $2Q = 2.67\,\text{mC}$, shared between two capacitors in parallel at a common voltage:

$$V_f = \frac{2Q}{C_1 + C_2} = \boxed{444\,\text{V}} \text{ across each},$$

$$Q_1 = C_1 V_f = \boxed{0.889\,\text{mC}}, \qquad Q_2 = C_2 V_f = \boxed{1.78\,\text{mC}}.$$

**(c)**

$$U_f = \tfrac{1}{2}(C_1 + C_2)V_f^2 = \boxed{0.593\,\mathrm{J}}, \qquad \frac{U_0 - U_f}{U_0} = \boxed{\tfrac{1}{9} \approx 11\%}.$$

(Exactly: $U_0 = \frac{1}{2}Q^2(1/C_1 + 1/C_2) = 3Q^2/8\,\mu\mathrm{F}^{-1}$ and $U_f = (2Q)^2/2(C_1 + C_2) = Q^2/3\,\mu\mathrm{F}^{-1}$, so $U_f/U_0 = 8/9$.) The missing 0.074 J is dissipated as heat in the connecting wires (plus a little electromagnetic radiation) while charge flows to equalise the voltages. A smaller resistance does *not* reduce the loss: it gives a larger current for a shorter time, but the energy dissipated is set only by the initial and final states.

**Problem 4.** **(a)** Between the conductors (Gauss) $E = Q/4\pi\varepsilon_0 r^2$, so

$$\Delta V = \int_a^b E\,dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \qquad \boxed{C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0 ab}{b-a}}.$$

**(b)**

$$\boxed{U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}.$$

**(c)**

$$U = \int_a^b \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2\,dr = \frac{Q^2}{8\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \checkmark$$

in agreement with (b).

**(d)** As $b \to \infty$, $\boxed{U = \dfrac{Q^2}{8\pi\varepsilon_0 a}}$ ($C = 4\pi\varepsilon_0 a$). The energy stored between $a$ and $r$ is proportional to $1/a - 1/r$, so the fraction within $r$ is $1 - a/r$:

$$\text{within } 2a: \ \boxed{1/2}, \qquad 1 - a/r = 0.9 \ \Rightarrow \ \boxed{r = 10a}.$$

Half the energy lies within one radius of the surface, even though the field extends to infinity.

**(e)** $Q = 4\pi\varepsilon_0 aV = 4\pi(8.854 \times 10^{-12})(0.15)(3.0 \times 10^5) = \boxed{5.0\,\mu\mathrm{C}}$, and $U = \frac{1}{2}QV = \boxed{0.75\,\mathrm{J}}$.