PHYS 216 – Assignment #3 – Solutions – Due Oct 14

Chapters 6–7: Gauss's Law, Conductors & Electric Potential

Throughout, $k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\,\mathrm{N\,m^2/C^2}$.

Problem 1. **(a)** Transferred charge: $q = Ne = (1.0 \times 10^{12})(1.602 \times 10^{-19}\,\mathrm{C}) = 1.60 \times 10^{-7}\,\mathrm{C}$. The left plate is left with $+q$, the right plate gains $-q$. The two charges attract, so (for large plates) all of the charge sits on the *inner* faces; the outer faces carry essentially none.

$$\sigma = \frac{q}{A} = \frac{1.60 \times 10^{-7}\,\mathrm{C}}{0.0400\,\mathrm{m^2}} = \boxed{4.0 \times 10^{-6}\,\mathrm{C/m^2}} \quad (+ \text{ on left, } - \text{ on right}).$$

(b) Between the plates $E = \sigma/\epsilon_0 = 4.5 \times 10^5\,\mathrm{N/C}$, directed from the left (positive) plate toward the right (negative) plate. Outside, the fields of the two sheets cancel: $E \approx 0$.

(c) $\Delta V = Ed = (4.52 \times 10^5\,\mathrm{N/C})(0.020\,\mathrm{m}) = \boxed{9.0 \times 10^3\,\mathrm{V}}$; the left (positive) plate is at the higher potential.

Problem 2. **(a)** Sketch should show: lines radiating from $+Q$, bunched on the side nearest the cavity wall, ending *perpendicular* to the inner surface; no lines in the metal; outside, lines leaving the outer surface perpendicularly and *radially symmetric about the centre of the sphere* (not about Q), with the same number of lines as leave Q.

(b)

- $r_1 < r < r_2$ (in the metal): $\vec{E} = 0$ (electrostatic equilibrium). A Gaussian sphere in the metal then has zero flux, so $q_{\mathrm{enc}} = 0$.
- $r > r_2$: The outer surface charge is uniform (see (c)), so the field has spherical symmetry about the centre. A Gaussian sphere of radius r encloses $Q + (-Q) + Q = Q$:

  $$\boxed{\vec{E} = \frac{kQ}{r^2}\,\hat{r}} \qquad (r > r_2),$$

  with $\hat{r}$ measured from the centre of the metal sphere, *independent of* d.
- $r < r_1$: Gauss's law is still *true*, but the charge distribution (point charge plus non-uniform induced charge) has no symmetry, so $|\vec{E}|$ is not constant on any simple surface and cannot be pulled out of the flux integral. Not feasible.

(c) From (b), $q_{\mathrm{inner}} = \boxed{-Q}$, non-uniform: denser on the part of the wall nearest Q, arranged so that the field of Q plus the inner charge vanishes everywhere in the metal and beyond. Since the sphere is neutral, $q_{\mathrm{outer}} = \boxed{+Q}$, *uniform*, $\sigma = Q/4\pi r_2^2$. The combination Q + inner charge produces zero field outside the cavity, so the outer charge "feels" only itself on an isolated spherical conductor, and the equilibrium distribution is uniform. The metal shields the outside from any information about where Q sits.

Problem 3. **(a)** Coaxial Gaussian cylinder of radius r, length L; by symmetry $\vec{E} = E(r)\hat{r}$, with no flux through the end caps.

$r < R$: $q_{\text{enc}} = \int_0^r \rho_0 \frac{r'}{R}\, 2\pi r' L\, dr' = \dfrac{2\pi\rho_0 L r^3}{3R}$, so

$$E(2\pi rL) = \frac{q_{\text{enc}}}{\epsilon_0} \;\Rightarrow\; \boxed{E = \frac{\rho_0 r^2}{3\epsilon_0 R}}.$$

$r > R$: $q_{\text{enc}} = \dfrac{2\pi\rho_0 L R^2}{3}$, so $\boxed{E = \dfrac{\rho_0 R^2}{3\epsilon_0 r}}$. The two agree at $r = R$. Outside, this is the field of a line charge $\lambda = 2\pi\rho_0 R^2/3$.

(b) Any of: a line charge $\lambda = 2\pi\rho_0 R^2/3$ on the axis; a uniform surface charge $\sigma = \lambda/2\pi R = \rho_0 R/3$ on the surface $r = R$; a uniform volume density $\rho_0' = 2\rho_0/3$ throughout $r < R$.

(c) The distribution must (i) be infinitely long, uniform along the axis, and depend only on r (cylindrical symmetry), and (ii) contain the same charge per unit length, $\lambda = 2\pi\rho_0 R^2/3$, within $r \le R$ (with none outside).

(d)

$$V(0) - V(R) = \int_0^R E\, dr = \frac{\rho_0}{3\epsilon_0 R}\int_0^R r^2\, dr = \boxed{\frac{\rho_0 R^2}{9\epsilon_0}},$$

with the axis at the higher potential ($\rho_0 > 0$). Outside, $V(r) - V(R) = -\frac{\lambda}{2\pi\epsilon_0}\ln(r/R)$, which diverges as $r \to \infty$: the charge itself extends to infinity, so $V(\infty)$ cannot serve as a finite reference.

Problem 4. **(a)** $\vec{E} = -\nabla V$:

$$\boxed{E_x = y^2 z - 4y, \qquad E_y = 2xyz - 4x, \qquad E_z = xy^2} \quad (\text{V/m}).$$

(b) At $(1,1,1)$: $\vec{E} = (-3\hat{\imath} - 2\hat{\jmath} + \hat{k})\,\text{V/m}$, $|\vec{E}| = \sqrt{14} = 3.7\,\text{V/m}$.

(c) $V(0,0,0) = 0$, $V(1,1,1) = -1 + 4 = 3\,\text{V}$. $W_{\text{ext}} = q\,\Delta V = (2.0\times10^{-6}\,\text{C})(3\,\text{V}) = \boxed{6.0\times10^{-6}\,\text{J}}$. Path-independent: $\vec{E}$ is the gradient of a scalar, so the electric force is conservative and the work depends only on the endpoints.

Problem 5. **(a)** $E = 0$ in the shell's metal, so a Gaussian sphere with $b < r < c$ encloses zero charge: $q_b = \boxed{-5.0\,\mu\text{C}}$. Then $q_c = -3.0 - (-5.0) = \boxed{+2.0\,\mu\text{C}}$.

(b) With $kq_a = 4.50\times10^4\,\text{N m}^2/\text{C}$ and $k(q_a + q_{\text{shell}}) = 1.80\times10^4\,\text{N m}^2/\text{C}$:

$$E(r) = \begin{cases} 0, & r < a \\ \dfrac{4.50\times10^4}{r^2}\ \text{N/C}, & a < r < b \quad (1.1\times10^8\,\text{N/C at } a,\ 1.8\times10^7\,\text{N/C at } b) \\ 0, & b < r < c \\ \dfrac{1.80\times10^4}{r^2}\ \text{N/C}, & r > c \quad (5.0\times10^6\,\text{N/C at } c) \end{cases}$$

all directed radially outward (r in metres).

(c) $V_{\text{shell}} = V(c) = \dfrac{k(q_a + q_{\text{shell}})}{c} = \boxed{3.0\times10^5\,\text{V}}$.

$$V_{\text{sphere}} = V_{\text{shell}} + kq_a\left(\frac{1}{a} - \frac{1}{b}\right) = 3.0\times10^5\,\text{V} + (4.50\times10^4)(50-20)\,\text{V} = \boxed{1.65\times10^6\,\text{V}}.$$

(d) V is continuous everywhere: constant (1.65 MV) for $r < a$, falling as $1/r$ (offset) to 0.30 MV at b, constant through the shell, then $\propto 1/r$ for $r > c$. E jumps at $r = a$, b, and c, the three charged surfaces, by σ/ϵ_0 (Gauss's law applied to a thin pillbox straddling each surface).

Problem 6. **(a)** Every element of a ring is the same distance $\sqrt{(z \mp a)^2 + R^2}$ from the axis point, so each ring contributes $kq/\text{distance}$:

$$\boxed{V(z) = kQ\left[\frac{1}{\sqrt{(z-a)^2+R^2}} - \frac{1}{\sqrt{(z+a)^2+R^2}}\right].}$$

(b) On the axis $E_\rho = 0$ by symmetry, and $E_z = -\partial V/\partial z$:

$$\boxed{E_z(z) = kQ\left[\frac{z-a}{[(z-a)^2+R^2]^{3/2}} - \frac{z+a}{[(z+a)^2+R^2]^{3/2}}\right].}$$

At $z = 0$: $E_z = -\dfrac{2kQa}{(a^2+R^2)^{3/2}}$. *Direct check:* the on-axis field of a ring at height h above the field point is $kqh/(h^2+R^2)^{3/2}$, directed away from a positive ring. At the origin, the $+Q$ ring (above) pushes in $-\hat{k}$, and the $-Q$ ring (below) pulls in $-\hat{k}$, each with magnitude $kQa/(a^2+R^2)^{3/2}$: $\vec{E}(0) = -\dfrac{2kQa}{(a^2+R^2)^{3/2}}\hat{k}$. ✓

(c) For $|z| \gg a, R$, $\left[(z \mp a)^2 + R^2\right]^{-1/2} \approx \frac{1}{|z|}\left(1 \pm \frac{a}{z} + \dots\right)$, so

$$V \approx \frac{2kQa}{z^2}\,\mathrm{sgn}(z) = \frac{kp\,z}{|z|^3}, \qquad E_z \approx \frac{2kp}{|z|^3}, \qquad \boxed{\vec{p} = 2Qa\,\hat{k}}$$

(pointing from $-Q$ to $+Q$). This is the on-axis point-dipole field from A2 Problem 2(b): $2kp/y^3 = Qa/\pi\epsilon_0 y^3$ with $p = 2Qa$. The ring radius R appears only in higher-order $(1/z^4)$ terms.

(d) $E_z = -\partial V/\partial z$ needs only how V changes *along* the axis, which $V(z)$ supplies. $E_\rho = -\partial V/\partial\rho$ needs how V changes *perpendicular* to the axis, which a function known only at $\rho = 0$ cannot give. On the axis itself $E_\rho = 0$: by rotational symmetry, every charge element's radial pull is cancelled by the element diametrically opposite.

(e) Bonus. For $R \ll a$, $(a^2+R^2)^{-3/2} = a^{-3}\left(1 + R^2/a^2\right)^{-3/2} \approx a^{-3}\left(1 - \frac{3R^2}{2a^2}\right)$:

$$\boxed{\vec{E}(0) \approx -\frac{2kQ}{a^2}\left(1 - \frac{3R^2}{2a^2}\right)\hat{k}.}$$

As $R \to 0$ this recovers two point charges $\pm Q$ at $\pm a$. Finite R *reduces* the field, since the ring's charge is farther away and partly pulls sideways (those components cancel).